Measure of phonon-number moments and motional quadratures through infinitesimal-time probing of trapped ions

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Abstract. A method for gaining information about the phonon-number moments and the generalized nonlinear and linear quadratures in the motion of trapped ions (in particular, position and momentum) is proposed, valid inside and outside the Lamb-Dicke regime. It is based on the measurement of first time derivatives of electronic populations, evaluated at the motion-probe interaction time $\tau = 0$. In contrast to other state-reconstruction proposals, based on measuring Rabi oscillations or dispersive interactions, the present scheme can be performed resonantly at infinitesimal short motion-probe interaction times, remaining thus insensitive to decoherence processes.

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1. Introduction

A single ion in a radio-frequency trap is an ideal system for the investigation of the basic and intriguing features of quantum mechanics [1]. Recent advances in the manipulation of the internal and external degrees of freedom of the trapped particle allowed, for example, the realization of laser cooling to the ground state for extended periods of time [2, 3], the creation of nonclassical motional states [4, 5], entanglement between external and internal degrees of freedom of one ion or several ions [6, 7, 8, 9, 10], quantum gates for quantum computation [11, 12, 13], and single photon emission on demand [14].

An important aspect at this advanced level of quantum state engineering is the ability to obtain as much information as possible, and with high efficiency, about the motional state of the trapped atom. Moreover, since the Lamb-Dicke (LD) regime, where the motion of the particle is restricted to a region smaller than a wavelength, is sometimes difficult to achieve in a trap, it is desirable to have measurement schemes working also outside this regime [15].

So far, several techniques have been proposed to gain information about the motional quantum state of a trapped particle. The central idea in all these methods is the mapping of the ionic external dynamics to an internal degree of freedom, where the latter can be read out, for example, with electron shelving techniques [16]. Some of these procedures are applicable only inside the LD regime [17, 18, 19, 20], whereas others work also beyond this limit [21, 22, 23, 24, 25, 26], allowing to derive in some cases the complete motional density operator. However, a typical requirement of conventional schemes is that the measurements have to be performed during relatively long interaction times, through several Rabi oscillations in the resonant case or slow phase shifts in the dispersive case, where decoherence mechanisms cannot be neglected.

In this paper, we present a method that allows to determine the expectation value of the phonon-number moments $\langle \hat{n}^p \rangle$ of a trapped ion for any positive integer p, as well as any generalized motional nonlinear quadrature $\frac{1}{2}\langle \hat{f}(\hat{n})\hat{a}e^{-i\phi} + \hat{a}^\dagger \hat{f}(\hat{n})e^{i\phi}\rangle$ ‡. Here, $\hat{a}$ and $\hat{a}^\dagger$ are the phonon annihilation and creation operators, respectively, $\hat{n} = \hat{a}^\dagger \hat{a}$, $\hat{f}(\hat{n})$ is a function of $\hat{n}$, and ϕ is an arbitrary phase factor. In particular, we are able to measure the expectation value of the linear quadratures, i.e., when $\hat{f}(\hat{n}) = 1$, including the ionic position and momentum. Our technique is not restricted to the LD regime and, moreover, can be extended easily to all spatial dimensions, or to systems containing more than one trapped ion. Furthermore, in contrast to the methods presented so far, our proposal allows to obtain the required information in an extremely short motion-probe interaction time, which is useful in particular in presence of strong decoherence processes.

In Sec. 2, we show for a single trapped ion how to obtain information about the expectation values of the phonon-number moments (Sec. 2.1) and the generalized motional nonlinear and linear quadratures (Sec. 2.2) through infinitesimal-time motion-probe interactions. In Sec. 3, we extend the method to the N-ion case. In Sec. 4, we summarize our central results.

2. Single ion case

2.1. Phonon-number moments

We consider a single two-level ion trapped in a harmonic potential. For the sake of simplicity, we confine our treatment to one motional degree of freedom, though all results are easily generalizable to three dimensions. The system is described by the Hamiltonian

$$\hat{H}_0 = \hbar\nu(\hat{n} + 1/2) + \hbar\omega_0 |e\rangle\langle e|, \tag{1}$$

where ν is the harmonic oscillator frequency, $|e\rangle$ and $|g\rangle$ are the electronic upper and lower states of the two-level ion, respectively, and ω_0 is the associated transition frequency.

The ion is excited by a travelling laser beam resonant with the carrier electronic transition, leaving unchanged the motional populations. In the usual rotating-wave approximation, and in the interaction picture, the Hamiltonian reads [15, 24]

$$\hat{H}_{\rm int} = \frac{1}{2}\hbar\Omega_L(\hat{\sigma}^+ + \hat{\sigma}^-)\hat{f}_0(\hat{n};\eta), \tag{2}$$

‡ Generalized nonlinear quadratures can be related to nonlinear coherent states, theoretically studied in [27]. A method for generating nonlinear coherent states in trapped ions, but not for measuring them, can be found in [28].

where Ω_L is the Rabi frequency, $\hat{\sigma}^{\pm}$ are the electronic two-level flip operators, η is the LD parameter ($\eta = k_x\langle x_0\rangle$, $\langle x_0\rangle$ being the extension of the ground state of the motional mode and k_x the projection of the laser wave vector on the trap axis), and $\hat{f}_0(\hat{n};\eta)$ is given by

$$\hat{f}_0(\hat{n};\eta) = e^{-\eta^2/2}\sum_{l=0}^{\infty}\frac{(i\eta)^{2l}}{l!^2}\frac{\hat{n}!}{(\hat{n}-l)!}. \tag{3}$$

At any time t, the probability of finding the ion in the excited state $|e\rangle$ is given by

$$P_e(t) = \mathrm{Tr}\Big[\hat{\rho}(t)|e\rangle\langle e|\Big] = \Big\langle |e\rangle\langle e|\Big\rangle, \tag{4}$$

where $\hat{\rho}(t)$ is the system density operator, describing the internal and external degrees of freedom of the trapped particle.

Considering that, for any operator $\hat{A}$,

$$\frac{d}{dt}\langle\hat{A}\rangle = \frac{1}{i\hbar}\Big\langle[\hat{A},\hat{H}]\Big\rangle + \Big\langle\frac{\partial\hat{A}}{\partial t}\Big\rangle, \tag{5}$$

we get, for $\hat{A} = |e\rangle\langle e|$,

$$\frac{dP_e}{dt} = \frac{1}{i\hbar}\Big\langle\Big[|e\rangle\langle e|,\hat{H}\Big]\Big\rangle. \tag{6}$$

In our case, we have $\big[|e\rangle\langle e|,\hat{H}_0\big] = 0$, so that

$$\frac{dP_e}{dt} = \frac{1}{i\hbar}\Big\langle\Big[|e\rangle\langle e|,\hat{H}_{\mathrm{int}}\Big]\Big\rangle. \tag{7}$$

Since, in the interaction picture,

$$\big[|e\rangle\langle e|,\hat{H}_{\mathrm{int}}\big] = \frac{\hbar\Omega_L}{2}(\hat{\sigma}^+ - \hat{\sigma}^-)\hat{f}_0(\hat{n},\eta), \tag{8}$$

we obtain, from (7) and (8),

$$\frac{dP_e}{dt} = \frac{\Omega_L}{2i}\mathrm{Tr}\big[\hat{\rho}(t)(\hat{\sigma}^+ - \hat{\sigma}^-)\hat{f}_0(\hat{n},\eta)\big]. \tag{9}$$

Next, we consider that the ion is prepared initially in the state

$$\hat{\rho}(0) = |\pm_\phi\rangle\langle\pm_\phi|\otimes\hat{\rho}_f, \tag{10}$$

where

$$|\pm_\phi\rangle = \frac{1}{\sqrt{2}}(|g\rangle \pm e^{i\phi}|e\rangle), \tag{11}$$

and $\hat{\rho}_f$ is the phonon state we aim to characterize. Then, a straightforward calculation yields

$$\mathrm{Tr}\big[\hat{\rho}(0)(\hat{\sigma}^+ - \hat{\sigma}^-)\hat{f}_0(\hat{n},\eta)\big] = \mp i\sin(\phi)\mathrm{Tr}\big[\hat{\rho}_f\hat{f}_0(\hat{n},\eta)\big], \tag{12}$$

and thus, using Eq. (9),

$$\left.\frac{dP_e^{\pm_\phi}}{d\tau}\right|_{\tau=0} = \mp\sin(\phi)\langle\hat{f}_0(\hat{n},\eta)\rangle, \tag{13}$$

where τ is the dimensionless time $\Omega_L t/2$ and the token $\pm_\phi$ stands for the two parameters (sign $\pm$ and phase ϕ) defining state $|\pm_\phi\rangle$ of Eq. (11).

Eq. (13) shows that the mean value of the nonlinear operator $\hat{f}_0(\hat{n};\eta)$ is determined by the time derivative of the population of the excited state at the initial interaction time $\tau = 0$. This allows to gain information about the ionic motional state, in particular, as will be shown below, about the phonon-number moments. Moreover, since the time derivative of the excitation probability $P_e^{\pm\phi}$ is evaluated at $\tau = 0$, this information can be obtained in a very short motion-probe interaction time, even before decoherence mechanisms can affect the initial coherent evolution. Remark further that the "contrast" in the measurement of $\langle \hat{f}_0(\hat{n},\eta)\rangle$, as seen in Eq. (13), can be tuned with the proper choice of the phase ϕ.

For small LD parameters, a series expansion of the nonlinear operator $\hat{f}_0(\hat{n};\eta)$ leads to the expression

$$\langle \hat{f}_0(\hat{n};\eta)\rangle \simeq 1 - \eta^2\langle \hat{n}\rangle + \frac{\eta^4}{4}\langle \hat{n}^2\rangle. \tag{14}$$

Thus, if we repeat the measurements with two known LD parameters, η_1 and η_2 (varying, e.g., the angle between the laser beam and the trap axis), we can derive $\langle \hat{n}\rangle$ and $\langle \hat{n}^2\rangle$ by solving the linear system

$$\begin{cases} \langle \hat{f}_0(\hat{n},\eta_1)\rangle & = 1 - \eta_1^2\langle \hat{n}\rangle + \frac{\eta_1^4}{4}\langle \hat{n}^2\rangle, \\ \langle \hat{f}_0(\hat{n},\eta_2)\rangle & = 1 - \eta_2^2\langle \hat{n}\rangle + \frac{\eta_2^4}{4}\langle \hat{n}^2\rangle, \end{cases} \tag{15}$$

from which, for example, the Fano-Mandel parameter [29]

$$Q = \frac{\langle \hat{n}^2\rangle - \langle \hat{n}\rangle^2}{\langle \hat{n}\rangle} \tag{16}$$

can be extracted. It is known that the Fano-Mandel parameter allows to determine the "classicality" of a given motional state, since quantum states with $Q > 1$ are considered as classical states, and those with $Q < 1$ as nonclassical states, due to their sub-Poissonian phonon-number distribution.

This low order moment determination scheme can be generalized to higher order vibronic moments $\langle \hat{n}^p\rangle$ ($p > 0$). Indeed, for higher LD parameters, the series expansion of $\hat{f}_0(\hat{n},\eta)$ in Eq. (3) will extend beyond the $\langle \hat{n}^2\rangle$ term, up to a certain moment $\langle \hat{n}^N\rangle$ §. We could repeat then the measurement technique, as in Eq. (15), for N known LD parameters, yielding a linear system from which the N first vibronic moments become accessible. Each $\langle \hat{n}^p\rangle$, $0 < p \leq N$, could also be determined by measuring only once the initial time derivative of $P_e^{\pm\phi}$. To show this, we require that the ion is submitted to N simultaneous laser interactions, each of them resonant with the electronic transition and leaving the motional state unchanged. In the interaction picture, they give rise to the simultaneous action of Hamiltonians

$$\hat{H}^j_{\rm int} = \frac{1}{2}\hbar\Omega_j\hat{\sigma}^+\hat{f}_0(\hat{n};\eta_j) + {\rm H.c.}, \quad j = 1,\ldots,N, \tag{17}$$

where Ω_j and η_j are, respectively, the electronic Rabi frequency and the LD parameter of laser j. The total Hamiltonian will be given in this case by

$$\hat{H}_{\rm int} = \sum_{j=1}^{N}\hat{H}^j_{\rm int} = \frac{\hbar\Omega_L}{2}(\hat{\sigma}^+ + \hat{\sigma}^-)\hat{F}_0(\hat{n}), \tag{18}$$

§ The series of Eq. (3) is convergent for all η. When the sum of this series is truncated at $l = N$, the coefficient of the $\hat{n}^N$ term is given by $(-1)^N e^{-\eta^2/2}\eta^{2N}/N!^2$. For any η value, this coefficient can be made arbitrarily small by a proper choice of N and the series may be truncated within good approximation up to this term.

where $\Omega_L = \max(\Omega_j)$ and

$$\hat{F}_0(\hat{n}) = \sum_{j=1}^{N} \frac{\Omega_j}{\Omega_L} \hat{f}_0(\hat{n};\eta_j). \tag{19}$$

If the system is again initially prepared in the state $|\pm_\phi\rangle\langle\pm_\phi|\otimes\hat{\rho}_f$, we get, similarly to Eq. (13),

$$\left.\frac{dP_e^{\pm_\phi}}{d\tau}\right|_{\tau=0} = \mp \sin(\phi)\langle\hat{F}_0(\hat{n})\rangle. \tag{20}$$

It has been shown by de Matos Filho and Vogel [24] that $\hat{F}_0(\hat{n})$ may be rewritten in the form of a Taylor series

$$\hat{F}_0(\hat{n}) = \sum_{p=0}^{\infty} c_p \hat{n}^p, \tag{21}$$

where the Taylor coefficients c_p are given by

$$c_p = \begin{cases} \displaystyle\sum_{j=1}^{N} e^{-\eta_j^2/2} \frac{\Omega_j}{\Omega_L}, & \text{if} \quad p = 0, \\ \displaystyle(-1)^p \sum_{j=1}^{N} e^{-\eta_j^2/2} \frac{\Omega_j}{\Omega_L} \left(\sum_{m=p}^{\infty} a_p^m \frac{\eta_j^{2m}}{m!^2} \right), & \text{if} \quad p \neq 0, \end{cases} \tag{22}$$

with

$$a_p^m = \begin{cases} 1, & \text{if} \quad p = m, \\ \displaystyle\sum_{j_{i_1} < j_{i_2} < \ldots < j_{i_{m-p}} = 1}^{m-1} j_{i_1} j_{i_2} \ldots j_{i_{m-p}} & \text{if} \quad p < m. \end{cases} \tag{23}$$

This means that for given values of the N LD parameters η_j (fixed by the geometry of the laser beams regarding to the trap axis up to the maximum value $\eta_{\max} = k\langle x_0\rangle$, k being the wave vector of the lasers), the c_p coefficients are linear combination of all Rabi frequencies Ω_j. In this way, the use of N lasers allows to fix at will N coefficients of the Taylor series. In particular, if the N first coefficients are set to 0, we obtain

$$\hat{F}_0(\hat{n}) = \mathcal{O}(\frac{\eta_{\max}^{2N}}{N!^2} \hat{n}^N). \tag{24}$$

Note that it is always possible to choose the N value such that $\mathcal{O}(\frac{\eta_{\max}^{2N}}{N!^2} \hat{n}^N)$ is negligible, even outside the LD regime where $\eta_{\max}$ can be greater than 1. The mean value of $\mathcal{O}(\frac{\eta_{\max}^{2N}}{N!^2} \hat{n}^N)$ can be verified experimentally by measuring the time derivative of the population of the excited state at interaction time $\tau = 0$. According to Eq. (20), the outcome of this measurement yields $\langle\mathcal{O}(\frac{\eta_{\max}^{2N}}{N!^2} \hat{n}^N)\rangle$ so that we can check if it is indeed negligible for the phonon distribution we aim to characterize (in this case it is recommendable to set $\phi = \pm\pi/2$).

The Rabi frequencies of the N lasers could also be chosen in such a manner that only a single coefficient c_p ($p < N$) is equal to 1, the others remaining equal to 0. In this case, according to Eq. (21) and considering that all terms $\hat{n}^r$, $r \geqslant N$, are negligible,

$$\hat{F}_0(\hat{n}) = \hat{n}^p, \tag{25}$$

and the measurement of $\left.\frac{dP_e^{\pm\phi}}{d\tau}\right|_{\tau=0}$ yields directly the vibronic moment $\langle \hat{n}^p \rangle$. This step can be reproduced for any $p < N$.

It is noteworthy to mention that the knowledge of the phonon-number moments $\langle \hat{n}^p \rangle$ for all positive integers p allows to derive the complete phonon distribution $p(n) \equiv \langle n|\hat{\rho}_f|n\rangle$, as known from classical statistics [30]. Also, the generalization to the $3D$ case provides measurement schemes for $\langle \hat{n}_x \rangle$, $\langle \hat{n}_y \rangle$, $\langle \hat{n}_z \rangle$, $\langle \hat{n}_x^2 \rangle$, $\langle \hat{n}_y^2 \rangle$, $\langle \hat{n}_z^2 \rangle$, $\langle \hat{n}_x \hat{n}_y \rangle$, $\langle \hat{n}_x \hat{n}_z \rangle$, $\langle \hat{n}_y \hat{n}_z \rangle$, $\langle \hat{n}_x \hat{n}_y \hat{n}_z \rangle$, and so on. Recently, the relevance of measuring different photon-number moments, through quantum field homodyning, for determining the "classicality" of arbitrary quantum states has been considered [31].

2.2. *Generalized nonlinear quadratures*

Next, we show that by using a red or blue sideband excitation, expectation values of generalized nonlinear quadrature moments, $\frac{1}{2}\langle \hat{f}(\hat{n})\hat{a}e^{-i\phi} + \hat{a}^\dagger \hat{f}(\hat{n})e^{i\phi}\rangle$, can be determined using similar techniques. In particular, we have access to the expectation values of the generalized *linear* quadrature moments, among them position $\langle \hat{x} \rangle$ and momentum $\langle \hat{p} \rangle$ ‖. As will be shown below, these expectation values can be obtained inside and outside of the LD regime.

Let us consider that the ion is excited with a red-sideband detuned laser (the blue-sideband case can be treated equally, leading to similar results). In the usual rotating-wave approximation, and in the interaction picture, the Hamiltonian reads [15, 24]

$$\hat{H}_{\rm int} = \frac{i}{2}\hbar\Omega_L \eta \hat{\sigma}^+ \hat{f}_1(\hat{n};\eta)\hat{a} + \text{H.c.}, \tag{26}$$

where

$$\hat{f}_1(\hat{n};\eta) = e^{-\eta^2/2}\sum_{l=0}^{\infty} \frac{(i\eta)^{2l}}{l!(l+1)!}\frac{\hat{n}!}{(\hat{n}-l)!} \quad . \tag{27}$$

If the ion system is initially prepared in the state $|\pm_\phi\rangle\langle\pm_\phi| \otimes \hat{\rho}_f$, we obtain straightforwardly from Eq. (6)

$$\left.\frac{dP_e^{\pm\phi}}{d\tau}\right|_{\tau=0} = \pm\frac{1}{2}\left\langle \hat{f}_1(\hat{n};\eta)\hat{a}e^{-i\phi} + \hat{a}^\dagger \hat{f}_1(\hat{n};\eta)e^{i\phi}\right\rangle, \tag{28}$$

where τ is the dimensionless time $\eta\Omega_L t/2$. In the LD regime, $\hat{f}_1(\hat{n};\eta) = 1$ and Eq. (28) reduces to the generalized quadrature $\hat{X}_\phi \equiv (\hat{a}e^{-i\phi} + \hat{a}^\dagger e^{i\phi})/2$,

$$\left.\frac{dP_e^{\pm\phi}}{d\tau}\right|_{\tau=0} = \pm\langle \hat{X}_\phi \rangle = \pm\frac{1}{2}\left(\cos(\phi)\frac{\langle \hat{x}\rangle}{\langle x_0\rangle} + \sin(\phi)\frac{\langle \hat{p}\rangle}{\langle p_0\rangle}\right), \tag{29}$$

where $\hat{x}$ and $\hat{p}$ are the trapped ion position and momentum, respectively, and $\langle x_0 \rangle$ and $\langle p_0 \rangle$ are the spreads of these quantities in the ground state of the trap potential. Eq. (29) shows that any quadrature moment $\hat{X}_\phi$ can be experimentally determined from the measurement of the initial time derivative of the population of the excited state, whereas, in particular, $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$ can be obtained by choosing $\phi = 0, \pi$ and $\phi = \pm\pi/2$, respectively.

‖ In this limit our results for trapped ions are analogous to the ones derived for measuring field quadratures in cavity QED setups: see [32].

Outside the LD regime, a similar information is gained using N simultaneous red-sideband detuned lasers. In the interaction picture, they give rise to the Hamiltonians

$$\hat{H}_{\rm int}^{j} = \frac{i}{2}\hbar\Omega_j\eta_j\hat{\sigma}^{+}\hat{f}_1(\hat{n};\eta_j)\hat{a} + {\rm H.c.}, \quad j = 1,\ldots,N, \tag{30}$$

where Ω_j and η_j are, respectively, the electronic Rabi frequency and the LD parameter of laser j. The total Hamiltonian is given in this case by

$$\hat{H}_{\rm int} = \sum_{j=1}^{N}\hat{H}_{\rm int}^{j} = \frac{i}{2}\hbar\Omega_L\hat{\sigma}^{+}\hat{F}_1(\hat{n})\hat{a} + {\rm H.c.}, \tag{31}$$

where $\Omega_L = \max(\Omega_j)$ and

$$\hat{F}_1(\hat{n}) = \sum_{j=1}^{N}\frac{\Omega_j\eta_j}{\Omega_L}\hat{f}_1(\hat{n};\eta_j). \tag{32}$$

If the ion is initially in the state $|\pm_\phi\rangle\langle\pm_\phi| \otimes \hat{\rho}_f$, we obtain, similarly to Eq. (28), with $\tau = \Omega_L t/2$,

$$\left.\frac{dP_e^{\pm_\phi}}{d\tau}\right|_{\tau=0} = \pm\frac{1}{2}\left\langle\hat{F}_1(\hat{n})\hat{a}e^{-i\phi} + \hat{a}^{\dagger}\hat{F}_1(\hat{n})e^{i\phi}\right\rangle. \tag{33}$$

According to de Matos Filho and Vogel [24], $\hat{F}_1(\hat{n})$ may be again rewritten in the form of a Taylor series

$$\hat{F}_1(\hat{n}) = \sum_{p=0}^{\infty} c_p\hat{n}^p, \tag{34}$$

where the c_p coefficients are linear combination of the N laser Rabi frequencies Ω_j. Similarly to the case of a resonant laser described in the preceding section, we can choose the N value in such a way that $\mathcal{O}(c_N\hat{n}^N)$ is negligible and fix next the N Rabi frequencies to set all coefficients c_p $(p < N)$ to 0, except c_0 to 1. In this case, $\hat{F}_1(\hat{n}) \simeq 1$, Eq. (33) reduces to Eq. (29) and the quadrature moments (and more specifically $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$) are determined as done in the case of the LD regime. Clearly, by following a similar procedure, we could also engineer $\hat{F}_1(\hat{n})$ to describe an arbitrary polynomial.

3. N-ion case

We now briefly discuss how our proposal can be generalized to a chain of N identical two-level ions in a linear Paul trap. This system is described by the Hamiltonian

$$\hat{H}_0 = \sum_{j=1}^{N}\hbar\nu_j(\hat{n}_j + 1/2) + \hbar\omega_0\sum_{k=1}^{N}|e\rangle_k\langle e|, \tag{35}$$

where ν_j are the frequencies associated with the collective motional modes, $\hat{n}_j = \hat{a}_j^{\dagger}\hat{a}_j$ are the respective phonon-number operators, $|e\rangle_k$ and $|g\rangle_k$ are the electronic states of the two-level ion k, and ω_0 is the electronic transition frequency of each ion.

The ions are illuminated by a laser beam resonant with their electronic transition while leaving unchanged the phonon population, realizing a nonlinear carrier

excitation. In the usual rotating-wave approximation and in the interaction picture, the Hamiltonian reads [15, 24]

$$\hat{H}_{\rm int} = \frac{1}{2}\hbar\Omega_L \sum_{k=1}^{N}(\hat{\sigma}_k^+ + \hat{\sigma}_k^-)\prod_{j=1}^{N}\hat{f}_0(\hat{n}_j;\eta_j), \tag{36}$$

where Ω_L is the electronic Rabi frequency, $\hat{\sigma}_k^{\pm}$ are the electronic two-level flip operators of atom k, η_j is the LD parameter related to the collective mode j ($\eta_j = k_x\langle x_0\rangle_j$, $\langle x_0\rangle_j$ being the extension of the ground state of mode j and k_x the projection of the laser wave vector on the trap axis), and

$$\hat{f}_0(\hat{n}_j;\eta_j) = e^{-\eta_j^2/2}\sum_{l=0}^{\infty}\frac{(i\eta_j)^{2l}}{l!^2}\frac{\hat{n}_j!}{(\hat{n}_j-l)!}. \tag{37}$$

In the following, we denote the product $\prod_{j=1}^{N}\hat{f}_0(\hat{n}_j;\eta_j)$ by $\hat{\mathcal{F}}_0$. The generalized relation

$$\frac{d(P_e)_k}{dt} = \frac{1}{i\hbar}\Big\langle\Big[|e\rangle_k\langle e|,\hat{H}\Big]\Big\rangle, \tag{38}$$

can be easily obtained, as in Eq. (6), where $(P_e)_k$ is the probability of finding ion k in its excited state. As $|e\rangle_k\langle e|$ commutes with $\hat{H}_0$ and any operator associated with other ions k', we get immediately

$$\frac{d(P_e)_k}{dt} = \frac{1}{i\hbar}\Big\langle\Big[|e\rangle_k\langle e|,\frac{1}{2}\hbar\Omega_L(\hat{\sigma}_k^+ + \hat{\sigma}_k^-)\hat{\mathcal{F}}_0\Big]\Big\rangle, \tag{39}$$

and thus

$$\frac{d(P_e)_k}{dt} = \frac{\Omega_L}{2i}{\rm Tr}\big[\hat{\rho}(t)(\hat{\sigma}_k^+ - \hat{\sigma}_k^-)\hat{\mathcal{F}}_0\big], \tag{40}$$

where $\hat{\rho}(t)$ is the N-ion system density operator.

Let us consider the following initial state

$$\hat{\rho}(0) = \hat{\rho}_k\otimes\hat{\rho}_A\otimes\hat{\rho}_f, \tag{41}$$

where

$$\hat{\rho}_k = |\pm_\phi\rangle_k\langle\pm_\phi| \tag{42}$$

is the electronic density operator of the k-th ion, with

$$|\pm_\phi\rangle_k = \frac{1}{\sqrt{2}}(|g\rangle_k \pm e^{i\phi}|e\rangle_k), \tag{43}$$

$\hat{\rho}_A$ is an arbitrary electronic density operator of the remaining ions, and $\hat{\rho}_f$ is the collective motional density operator associated with the N eigenmodes. Following similar steps as in the previous sections, we obtain

$${\rm Tr}\big[\hat{\rho}(0)(\hat{\sigma}_k^+ - \hat{\sigma}_k^-)\hat{\mathcal{F}}_0\big] = \mp i\sin(\phi){\rm Tr}\big[\hat{\rho}_f\hat{\mathcal{F}}_0\big], \tag{44}$$

similar to Eq. (12), and thus, using Eq. (40),

$$\left.\frac{d(P_e)_k}{d\tau}\right|_{\tau=0} = \mp\sin(\phi)\langle\hat{\mathcal{F}}_0\rangle, \tag{45}$$

where τ is the dimensionless time $\tau = \Omega_L t/2$. Eq. (45) tells us that by measuring the time derivative of the population of the excited state of *one* ion at $\tau = 0$, we are able to gain information about the *collective* nonlinear operator $\hat{\mathcal{F}}_0$.

4. Summary

In conclusion, we have proposed a method that allows to determine the phonon-number moments and the motional nonlinear and linear quadratures of a trapped ion, in particular the ionic position and momentum. Our method makes use of the nonlinear behavior of the ion-laser interaction in harmonic traps. The measurement of the phonon-number moments $\langle \hat{n}^p \rangle$ requires resonant carrier interaction, with no phonon gain or loss, while the measurement of the nonlinear quadrature moments $\frac{1}{2}\langle \hat{f}(\hat{n})\hat{a}e^{-i\phi} + \hat{a}^\dagger \hat{f}(\hat{n})e^{i\phi} \rangle$ demands the use of red or blue sideband excitations. In contrast to methods presented so far, our proposal is designed for measurements realized in very short probe laser interaction times, thus preventing the noisy action of decoherence processes. In addition, we have shown that our scheme works inside and outside the LD regime, and that it can be generalized to the case of N ions. In the latter case, information about a collective property is gained by the measurement of the excited state population of a single ion.

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